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Orbit depths of affine Kac–Moody algebras

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Abstract. Since the number of weights in a highest weight representation of an affine algebra is infinite, it is particularly useful for these algebras to classify weight systems in terms of Weyl orbits. It is shown that even though depths in affine orbits are unbounded, any weight depth may be determined in a number of steps that is the same for all weights. The concepts of root layers and families and partner roots are introduced and used in the procedure. A projection of weight vectors onto a Euclidean subspace is used to present a simple geometrical picture of orbits.

1. Introduction

In this paper we are concerned with two types of irreducible representations ('irreps') of affine Kac–Moody algebras, highest weight irreps and the adjoint (root) representation. It is useful to group the weights of a highest weight irrep in Weyl orbits, where the weights of an orbit are obtainable from each other by sequences of Weyl reflections. A useful concept for both finite and affine algebras is the depth of a weight in an orbit, defined as the minimum number of Weyl reflections associated with simple roots necessary to transform the weight into the dominant orbit weight. A measure of depth developed previously for finite algebras is shown to apply to affine algebras as well [1]. This measure is particularly useful for affine algebras, since the depths are unbounded while their determination requires a fixed number of steps that is the same for all weights.

In section 3, the concepts of root layers and families and of partner roots are introduced. A family is infinite, and includes one root from each layer. The number of roots in a layer is finite, and may be obtained from a simple formula. These concepts are used in section 4 both in applying the depth determination procedure, and in proving its validity.

2. Basic properties of affine algebras

The basic properties summarised in this section can be found in various references [2–4]. A simple affine algebra of rank $(n-1)$ is represented by an indecomposable Coxeter–Dynkin diagram with n vertices, numbered here from 1 to n. Each vertex represents a simple root R_i. The generalised, $n \times n$ Cartan matrix A is defined by the scalar product equation

$$A_{ij} = \langle R_i, R_j \rangle (2/R_j^2). \tag{2.1}$$

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The n Dynkin components of a weight M are denoted by m_i and defined by

$$m_i = \langle M, R_i \rangle (2/R_i^2). \tag{2.2}$$

One may specify a weight of a highest weight irrep completely by the n Dynkin components and one additional component m_0. This component is defined by $m_0 = -P_h$, where the displacement P_h for a weight M is the number of times the simple root R_h is subtracted from the highest weight to obtain M. For a root α, P_h is the negative of the number of times R_h is added to the zero root to obtain α. In [3] a particular simple root is chosen for R_h for each algebra, but I use here the generalisation of [4], in which any of the n simple roots may be identified with R_h.

The determinant of A is zero for affine algebras. There is a null vector δ that satisfies the equation $\delta A = 0$, where δ is a row vector. Let δ be written as a linear combination of simple roots, i.e.

$$\delta = \sum_i c_i R_i. \tag{2.3}$$

The vector δ is normalised by the condition that the c_i are as small as possible, consistent with all being positive integers. These c_i are called marks. The co-root $\alpha_i^\vee$ corresponding to the real non-zero root α_i is defined by the equation

$$\alpha_i^\vee = \alpha_i (2/\alpha_i^2). \tag{2.4}$$

The co-marks $c_i^\vee$ are defined by writing δ as a linear combination of the simple co-roots, i.e.

$$\delta = \sum_i c_i^\vee R_i^\vee. \tag{2.5}$$

Thus, $c_i^\vee = c_i R_i^2/2$.

The ratios of the lengths of the simple roots are given by the Coxeter–Dynkin diagram, or the Cartan matrix. The overall root normalisation is specified by the condition that the roots are as short as possible, consistent with the co-marks all being integers. With this convention the twist k of an algebra is given in terms of the length of the longest simple root R_l by the formula $k = \frac{1}{2} R_l^2$. The Coxeter–Dynkin diagrams for all the simple affine algebras are given in various references [2–4].

The level L of a weight M is defined by the scalar product,

$$L(M) = \langle M, \delta \rangle = \sum_i m_i c_i^\vee. \tag{2.6}$$

The level of all roots is zero, and the level of all weights in an irrep is the same. A dominant weight is one such that $m_i \geq 0$ for all i. The highest weight of a highest weight irrep is a dominant weight.

The symmetric matrix S is defined in terms of the co-roots by $S_{ij} = \langle R_i^\vee, R_j^\vee \rangle$. This matrix is related to the generalised Cartan matrix A by the equation $S = 2(R^2)^{-1} A$, where R^2 is the diagonal, positive-definite matrix with elements $(R^2)_{ij} = \delta_{ij} R_i^2$.

The finite subalgebra $\mathcal{S}_h$ (called here a basic subalgebra) is obtained by ignoring the simple root R_h of the affine algebra, i.e., by ignoring the h vertex and connecting lines in the diagram. The subweight with respect to S_h of an affine weight M is denoted by $M^{(h)}$. The $(n-1)$ Dynkin components of $M^{(h)}$ are obtained by ignoring the component m_h of the affine weight. The matrix $S^{(h)}$ is an $(n-1)$-dimensional square matrix obtained by ignoring the h row and column of S. The matrix $S^{(h)}$ does have an inverse, denoted by $G^{(h)}$. The $(n-1)$ dual components $M_i^{\vee(h)}$ of $M^{(h)}$ are given by the equation,

$$M_i^{\vee(h)} = \sum_j G_{ij}^{(h)} m_j. \tag{2.7}$$

I define an n-component vector $M^{\vee(h)*}$ by the equations

$$M_h^{\vee(h)*} = 0 \qquad M_i^{\vee(h)*} = M_i^{\vee(h)} \qquad \text{for } i \neq h. \tag{2.8}$$

Once the special root R_h is chosen two dual sets of $(n+1)$ components each may be defined. The extended Dynkin set is that discussed earlier, i.e., $(m_0 = -P_h, m_1 ... m_n)$. The dual components are

$$M_0^{\vee} = L(M)/c_h \tag{2.9a}$$

$$M_i^{\vee} = -P_h(c_i^{\vee}/c_h) + M_i^{\vee(h)*} \qquad \text{for } i \neq 0. \tag{2.9b}$$

The generalised affine scalar product of two vectors may then be written

$$\langle M, Q\rangle = \sum_{i=0}^{n} m_i Q_i^{\vee}. \tag{2.10}$$

If one makes use of the level equation for M (equation (2.6)) the scalar product may be written in the form

$$\langle M, Q\rangle = -[P_h(M)L(Q) + P_h(Q)L(M)]c_h^{-1} + \langle M, Q\rangle_h \tag{2.11}$$

where $\langle M, Q\rangle_h$ is the scalar product of the $(n-1)$-component subweights $M^{(h)}$ and $Q^{(h)}$.

If $L(Q) = 0$, it is seen from equation (2.9a) that $Q_0^{\vee} = 0$. In this case the weight may be written as a linear combination of the co-roots, and the coefficients are the dual components $Q_i^{\vee} (i \neq 0)$. Thus

$$Q = \sum_{i=1}^{n} Q_i^{\vee} R_i^{\vee}. \tag{2.12}$$

The Weyl reflection $W_\alpha(M)$ of a weight M associated with a non-zero root α is defined by

$$W_\alpha(M) = M - \langle M, \alpha\rangle(2/\alpha^2)\alpha. \tag{2.13}$$

If α is the simple root R_j, the reflection is denoted by W_j and is called simple. It follows from equations (2.13) and (2.2) that the reflected weight is

$$W_j(M) = M - m_j R_j. \tag{2.14}$$

The Weyl group consists of all sequences of zero or more Weyl reflections, and may be generated by the simple reflections alone. All weights related to a weight M by sequences of Weyl reflections comprise the Weyl orbit of M. If $L(M) > 0$, the highest weight M^{++} of the orbit is dominant, and is the only dominant weight of the orbit.

It is seen from equation (2.4) that the simple Weyl reflection W_j leads to a more negative weight, more positive weight, or the same weight if the Dynkin component m_j is positive, negative, or zero, respectively. A positive simple reflection series is a series of simple Weyl reflections, each of which leads to a more positive weight.

3. The adjoint representation

One may construct the positive roots by adding simple roots, using the same algorithm known for finite algebras. However, in the case of affine algebras the number of roots is infinite. The weights $m\delta$ are all roots, for all integer values of m. If m is positive or negative these roots are called imaginary roots; all other roots are called real roots.

A second method of obtaining the positive roots is to apply positive simple reflection series to the simple roots. This method yields all the real positive roots, but not the imaginary roots.

The root systems for all affine algebras are known to have the following properties (propositions 6.3d and 5.1a of reference [3]):

$$\text{(i) If } \alpha \text{ is a root, then } \alpha + lk\delta \text{ is a root} \tag{3.1}$$

for all integers l. Here k is the twist index, defined in section 2. (The possible values of k for the simple affine algebras are 1, 2, and 3.)

(ii) The multiplicities of all non-zero real roots are one.

It follows from equations (2.9*a*) and (2.9*b*) that any weight of level zero may be specified completely by its n co-root basis components $Q_i^\vee$.

The symbol $A \succ B$ for two weights on the same level means that the root-basis (or co-root basis) components of $A - B$ are non-negative, and at least one is positive. Several further properties of the root systems follow from combining equation (3.1) with the definition of simple roots. First, each real root belongs to a specific layer. For the roots α of the mth layer $\alpha \succ (m-1)k\delta$ and $mk\delta \succ \alpha$. A real root α may be written in the form

$$\alpha = \alpha_{1i} + (m-1)k\delta. \tag{3.2}$$

where α_{1i} is a root of the first layer, and the layer number m ranges through all the integers. The positive real roots correspond to positive values of m. Clearly, classifying by layer is different from classifying by a displacement P_h. The term family is used here to describe the roots corresponding to one α_{1i} and all values of m. For every real root α there is a different 'partner' root, defined to be the root in the layer of α and the family of $(-\alpha)$. Thus, the partner of the root of equation (3.2) is $mk\delta - \alpha_{1i}$.

An overlong root is defined to be a root with norm (length-squared) greater than 2; these exist only for the twisted ($k = 2$ or 3) algebras. It is known that if the k in the recursion relation of equation (3.1) were replaced by 1, the relation would still apply to all roots except the overlong roots (proposition 6.3 of reference [3]).

It can be shown that for all the simple affine algebras the number N_α of real roots in a layer is given by the formula

$$N_\alpha = k\,d\,C \tag{3.3}$$

where d is the rank ($d = n - 1$) and C is the Coxeter number, defined by

$$C = \sum_{i=1}^{n} c_i. \tag{3.4}$$

4. The depth measure

For finite algebras the depth of a weight M in an orbit has been defined as the minimum number of simple Weyl reflections necessary to transform M to the dominant orbit weight M^{++}; this is the number of terms in any positive simple reflection series from M to M^{++}. This definition is also appropriate for highest weight representations of affine algebras. Let S be a set of roots and M a weight of positive level. We denote by $S[M]$ the number of roots α in S that are obtusely inclined to M, i.e., satisfy the relation

$$\langle M, \alpha \rangle < 0. \tag{4.1}$$

It has been shown for finite algebras that the depth is equal to $\Pi[M]$, where Π is the set of all positive roots. This statement is true as well for highest weight representations of affine algebras. The proof will be given later. Here, we will assume the result, and show how the depth of any weight may be determined in a small number of steps.

When calculating a scalar product of the type of equation (4.1), it is convenient to use the co-root components $\alpha_i^\vee$ of the root α, determined from equation (2.12). Then, since the roots are of level zero, it is seen from equation (2.9a) that the 0 component does not contribute in equation (2.10), i.e.

$$\langle M, \alpha \rangle = \sum_{i=1}^{n} m_i \alpha_i^\vee. \tag{4.2}$$

The scalar product depends only on the Dynkin components of M.

We write the set Π as a union of disjoint subsets

$$\Pi = \sum_i (\pi_i + \pi_i') \tag{4.3}$$

where π_i is the set of positive roots in the family of α_{1i}, α_{1i} and α_{1i}' are partners, and the finite sum over i includes one of each pair of partner families. If the root α is written in the form of equation (3.2), it follows from equation (2.6) that

$$\langle M, \alpha \rangle = \langle M, \alpha_{1i} \rangle + (m-1)kL \tag{4.4}$$

where L is $L(M)$. Consequently, one can determine $\pi_i[M]$ from the quantity $X_i = -\langle M, \alpha_{1i} \rangle$. If a is a real number and b is a positive number, the symbol $(a/b)^+$ denotes the smallest non-negative integer n that satisfies $n \geq (a/b)$. (Thus, $n = 0$ if $a < 0$.) It follows from equation (4.4) that

$$\pi_i[M] = (X_i/kL)^+ \qquad \pi_i'[M] = (X_i'/kL)^+. \tag{4.5}$$

Since $\alpha_{1i} + \alpha_{1i}' = k\delta$, it follows that

$$X_i' = \langle M, \alpha_i \rangle - kL \tag{4.6}$$

so that the one scalar product $\langle M, \alpha_{1i} \rangle$ determines both $\pi_i[M]$ and $\pi_i'[M]$.

If the algebra is twisted ($k > 1$), the procedure may be shortened further. First we consider the overlong roots, the roots such that $\alpha^2 > 2$. It can be shown that the co-root-basis elements of these roots are integral multiples of k. Therefore, it is convenient to define the reduced root $\alpha^r_{1i} = \alpha_{1i}/k$, and to let $X_i = -\langle M, \alpha^r_{1i}\rangle$. With this redefinition, one uses equations (4.5) and (4.6) with the k factors replaced by unity.

Next, we consider the roots that are not overlong. These roots satisfy the recursion formula of equation (3.1) with k replaced by 1. Each layer consists of k mini-layers; for each root α of the first mini-layer, $\delta \succ \alpha$. Therefore, we may restrict the sum in equation (4.3) so that the α_{1i} are half the roots of the first mini-layer, and redefine α'_{1i} to be a mini-partner, defined by $\alpha_{1i} + \alpha'_{1i} = \delta$. Then π_i is redefined to be the set of k families related by multiples of δ, and the factors k in equations (4.5) and (4.6) should again be replaced by unity.

Figure 1. Coxeter–Dynkin diagram for $D_4^{(3)}$.

I illustrate this procedure with a weight of the $k = 3$ algebra $D_4^{(3)}$. The Coxeter–Dynkin diagram for this algebra is given in figure 1. If the roots in this diagram are numbered from left to right, the Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$

The marks are 1, 2, and 1 and the co-marks are $\{123\}$. (Curly brackets are used for co-root-basis components.) The level of a weight M is $L(M) = m_1 + 2m_2 + 3m_3$. The norms of the simple roots are 2, 2, and 6, respectively. There are six overlong roots in the first layer, $\{003\}$, $\{033\}$, $\{036\}$, and their respective partners $\{366\}$, $\{336\}$, and $\{333\}$. We need consider only the first three; the corresponding reduced roots α^r are $\{001\}$, $\{011\}$, and $\{012\}$.

There are 18 normal (norm 2) roots in the first layer, 6 in each mini-layer. The roots of the first mini-layer are $\{100\}$, $\{010\}$, $\{110\}$, and their mini-partners $\{023\}$, $\{113\}$, and $\{013\}$. Again we need consider only the first three.

Consider the weight M with Dynkin components $(-5\ -4\ 5)$. It is seen from equation (2.6) that the level of the weight is 2. Choosing the three reduced overlong and the three normal roots discussed above for α^r_{1i} ($\alpha^r_{1i} = \alpha_{1i}$ for the normal roots), one makes a table, shown in table 1. If $X_i \geq 0$, it is not necessary to compute X'_i. The depth of M is the sum of the numbers in the last column, i.e., 14.

One usually wants to know the dominant orbit weight M^{++} as well as the depth. For $D_4^{(3)}$ there are only two dominant level-2 weights, (200) and (010). Since the Dynkin components of M are not all divisible by 2, M^{++} cannot be (200) and so must be (010). General methods for finding M^{++} are given in [4]. For any h the difference in displacement P_h between M and M^{++} is given by the formula [4] $\Delta P_h = (c_h/2L)\Delta N_h$ where $N_h(M)$ is $\langle M, M\rangle_h$. In this case the displacement changes ΔP_i ($i = 1$ to 3) are 15, 25, and 10.

In some cases one can take advantage of simple bases. If the algebra is untwisted and the subalgebra S_h is fundamental (i.e., $c_h = 1$), then $\alpha^\vee_h = 0$ for one of every pair

Table 1. Contributions to depth of weight $(-5\ -4\ 5)$ of $D_4^{(3)}$.

α_{1i}^r	X_i	$X_i' = -X_i - 2$	Contribution to depth
{001}	−5	3	2
{011}	−1	−1	0
{012}	−6	4	2
{100}	5	—	3
{010}	4	—	2
{012}	9	—	5

of partner roots in the first layer. The set of these roots is the set of positive roots of $\mathcal{S}_h$. One may use these roots to compute the depth; the scalar product $\langle M, \alpha_{1i}\rangle$ is then $\langle M, \alpha_{1i}\rangle_h$. (This follows from equation (2.11), since P_h for these roots is 0.) In some cases simple orthogonal bases exist for $\mathcal{S}_h$; such bases are listed by King and Al-Qubanchi (reference [6], table 2).

Now I will show that the depth criterion involving equation (4.1) is valid for affine algebras as well as finite algebras. First, we review the proof for finite algebras [1]. We consider an arbitrary non-dominant weight M, and a simple Weyl reflection W_j associated with a negative Dynkin component m_j. Since the theorem is valid when $\Pi[M] = 0$, it is sufficient to show that $\Pi[W_j(M)] = \Pi[M] - 1$. It follows from equation (2.2) that $\langle M, R_j\rangle < 0$. Since the jth Dynkin component of R_j is equal to 2, it follows from equation (2.14) that $W_j(M)_j = -m_j$, so $\langle W_j(M), R_j\rangle > 0$. The finite set Π_j is defined to be the set of all positive roots except R_j. Since the R_j scalar product produces the desired decrease in $\Pi[M]$ (as one proceeds from M to $W_j(M)$), it is sufficient if $\Pi_j[M] = \Pi_j[W_j(M)]$. However, for a Weyl reflection, $\langle W_j(M), \alpha\rangle = \langle M, W_j(\alpha)\rangle$. Therefore, it is sufficient to show that

$$\Pi_j[M] = W_j(\Pi_j)[M]. \tag{4.7}$$

A lemma given by Jacobson states that if α is any positive root other than R_j, $W_j(\alpha)$ is also a positive root [7]. Hence $W_j(\Pi_j)$ is a set of different positive roots and does not contain R_j. Therefore $W_j(\Pi_j) \equiv \Pi_j$, and equation(4.7) is satisfied.

We next consider one of the proofs of Jacobson's lemma, one that is convenient for showing that the lemma applies to affine algebras. The roots α and $W_j(\alpha)$ are opposite members of an R_j root chain; that is, if β and γ are the more positive and more negative of α and $W_j(\alpha)$, respectively, the weights $\gamma, \gamma + R_j, \gamma + 2R_j, \ldots \beta$ are all roots. If γ is negative and β positive, the root chain must contain a zero root; otherwise the chain would contain two successive roots γ_1 and γ_2 such that γ_1 is negative, γ_2 is positive and $\gamma_2 = \gamma_1 + R_j$. This is impossible because the simple root R_j cannot be the sum of the two positive roots γ_2 and $-\gamma_1$. The only R_j root chain that contains a zero root is $-R_j, 0, R_j$. This follows because if α is a real root of an affine algebra (or of a finite algebra), 2α is not a root (proposition 5.1 of reference [3]). Therefore, the lemma is valid for affine algebras.

We next combine the lemma with the recursion property of equation (3.1). Since, for any integer n, the Dynkin components of $\alpha + nk\delta$ are the same as those of α, it follows from equation (2.14) that if $W_j(\alpha) = \beta$, then $W_j(\alpha + nk\delta) = \beta + nk\delta$. We consider the positive roots of the mth layer. The simple reflection W_j takes the root $(m-1)k\delta + R_j$ to $(m-1)k\delta - R_j$; the layer number is decreased by one. In a similar fashion W_j takes the partner root $mk\delta - R_j$ to $mk\delta + R_j$; the layer number is increased by one. For all real roots α of the mth layer except these two, $W_j(\alpha)$ is also in the mth layer.

In order to extend the finite-algebra proof to affine algebras, we need to find an appropriate finite root set Π_j. It is clear from equation (4.4) that for any weight M there is a layer number q such that no root of layer higher than the q layer can satisfy equation (4.1). We start by considering all the positive roots of the layers 1 through q. As before, we exclude the root R_j from this set. We then add to the set one root of the $(q+1)$ layer, the root $qk\delta + R_j$. The resulting set is Π_j. As before, the set $W_j(\Pi_j)$ is identical to the set Π_j, so the proof is valid.

5. A geometric picture

The signature of a positive root for a weight M is defined as negative if the inequality of equation (4.1) is satisfied; otherwise the signature is positive. Each weight may be classified by listing the roots with negative signatures. The number of roots in the list is the depth. In [1] several properties of this classification scheme were discussed. One can picture the weights geometrically for finite algebras, because the space is Euclidean.

In the case of affine algebras, the standard scalar product involves a weight space that is not Euclidean. In order to picture the effects of Weyl reflections, we will consider a particular projection of positive-level weights onto a space that is Euclidean.

If $\mathcal{A}$ is an affine algebra with n vertices, a Euclidean space of $(n-1)$ dimensions is defined by the simple roots corresponding to any of the basic subalgebras $\mathcal{S}_k$. For an affine weight M the vector in this space is defined by the Dynkin components other than m_k. If the affine vector is of level zero, the resulting $(n-1)$-component vector is the same no matter which of the n basic subalgebras is chosen [4]. Therefore, it is convenient to use all n Dynkin components to describe such a vector.

Given a weight M of positive level, I define a level-0 projection M_p by the equation

$$M_p = M - D \tag{5.1}$$

where the n Dynkin components d_i of D are

$$d_i = \frac{2L}{R_i^2 C} \tag{5.2}$$

and C is the Coxeter number, equation (3.4). It is easy to see that the level of D is L so that the level of M_p is zero. The Dynkin components of M_p are not integers, in general. Since the levels of all weights in an orbit are the same, the Weyl reflection of a projected weight may be defined by

$$W_\alpha(M_p) = W_\alpha(M) - D. \tag{5.3}$$

It follows from equations (5.1), (5.3), and the Weyl reflection equation, equation (2.13), that $W_\alpha(M_p) - M_p$ is a vector in the direction of α in the projection space. The component in the direction of α of any vector V is given by $\langle V, \alpha\rangle/|\alpha|$. I denote the average of the α components of M_p and $W_\alpha(M_p)$ by F_p, i.e., $F_p = [M_p + W_\alpha(M_p)]/(2|\alpha|)$. It follows from equations (5.1), (5.3) and (2.13) that

$$F_p = -\langle D, \alpha\rangle/|\alpha|. \tag{5.4}$$

Since this quantity is the same for all weights in an orbit, the transformation W_α does act like a reflection in the projection space. The reflection surface is of dimension $(n-2)$, is perpendicular to α and is displaced from the origin by the F_p of equation (5.4). A positive value of $\langle M_p, \alpha\rangle$ means that the projected weight M_p and the origin are on the same side of the reflection surface.

Let the root α be of the form $\alpha = R_i + l\delta$, where R_i is a simple root and l is an integer. It follows from equations (2.2), (2.6), (5.2) and (5.4) that

$$F_p = -\frac{L}{|R_i|}\left(\frac{1}{C} + l\right). \tag{5.5}$$

The simple roots correspond to $l = 0$ in equation (5.5). The finite volume enclosed by the n surfaces of the simple roots is the dominant chamber and includes the origin in the projection space. Each Weyl chamber is the same size and shape.

It is instructive to compare this picture with the corresponding picture for a finite algebra of rank n. In the finite case, the Weyl surfaces are planar, of dimension $(n-1)$, and pass through the origin. However, all weights in an orbit have the same length $\mathcal{L}$, so we consider a sphere centred at the origin, of radius $\mathcal{L}$. I use the word chambers here to refer to segments of this sphere bounded by intersections of the Weyl surfaces with the sphere. These intersections are of dimension $(n-2)$.

The projection-space chambers for affine algebras also represent the intersections of the Weyl sectors with a surface appropriate for a certain class of weights, weights of a particular level. The surface for affine weights is flat rather than spherical, so there are an infinite number of chambers.

A simple example is the algebra affine A_2, where the three simple roots lie in a plane, each of length $\sqrt{2}$ and at angles of 120° to the other two. The projection space is two dimensional and the chambers are equilateral triangles of altitude $L/\sqrt{2}$.

In this projection space generalisations of the theorems of reference [1] are valid. The allowed signature lists for the weights of an orbit are the same for all orbits of a given pattern, where a pattern is specified by identifying the Dynkin components of the dominant weight that are zero. In every orbit, there is exactly one weight either in or on a boundary of every chamber. If the weights are not on any boundaries, the orbit is maximal; in this case $m_i^{++} > 0$ for all i.

For each Weyl chamber V there is a unique member of the Weyl group that transforms V into the dominant chamber V^{++}; the chamber may be identified with this transformation. If two chambers differ in the signature of only one positive root, they intersect in an $(n-2)$-dimensional boundary and are called adjacent. If V corresponds to the Weyl transformation W, the chamber that corresponds to W^{-1} is called V^{-1}. As in the finite case, two chambers U and V are related by one simple Weyl reflection if and only if the inverse chambers U^{-1} and V^{-1} are adjacent. The other theorems of [1], concerning the signature condition and the orbit rule, are also valid for the affine case†

In conclusion, even though there are an infinite number of members of the Weyl group for an affine algebra, the depth of every weight in a highest weight irrep is finite, and this allows one to extend the theorems and proofs from the finite to affine algebras.

† If one uses or proves the signature condition of [1] in the affine case, one must consider the positive real *and imaginary* roots, with the signatures of the imaginary roots assigned to be positive.

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